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Three laterally coupled quantum waveguides: breaking of symmetry and resonance asymptotics

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Abstract
We consider the system of three quantum waveguides coupled laterally through small windows, and we study the breaking of geometrical symmetry. We investigate the behaviour of the resonance asymptotics and transition 'eigenvalue-resonance'.

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1. Introduction

The problem of coupled waveguides has attracted a new wave of interest recently due to the development of nanoelectronics. The problem of creation of new devices based on quantum interference cannot be solved without a theoretical description of different quantum systems. Recently, experimentalists began to produce a new class of objects: quantum dots and antidots, quantum wires (waveguides), etc. These systems are usually called mesoscopic systems because they are sufficiently large to be created experimentally, but sufficiently small to demonstrate the quantum character of an electron. In these semiconductor devices, the mean free path of the electron may be larger than the size of the system. In this situation, physicists usually deal with a ballistic regime. The description of ballistic electron transport in many mesoscopic quantum systems reduces to the description of electron wave propagation in a system of waveguides or layers, see, for example, [1–5]. The problem of bound states for laterally coupled waveguides has recently attracted a new wave of interest.

In this paper we deal with systems of two-dimensional waveguides coupled through small apertures. Let Ω_+, Ω_- be two waveguides of widths d_+, d_- coupled laterally through a small window of width $2a$. It has been proven in [6] that the Dirichlet Laplacian for this system has an eigenvalue λ_a close to the threshold. It can be estimated as

$$c_1 a^4 \leqslant \frac{\pi^2}{d_+^2} - \lambda_a \leqslant c_2 a^4 \tag{1}$$

for sufficiently small a (the order of this term was found in [7] on a physical level of rigor). Here c_1, c_2 are some constants, $d_+ > d_-$. The authors used a variational technique and

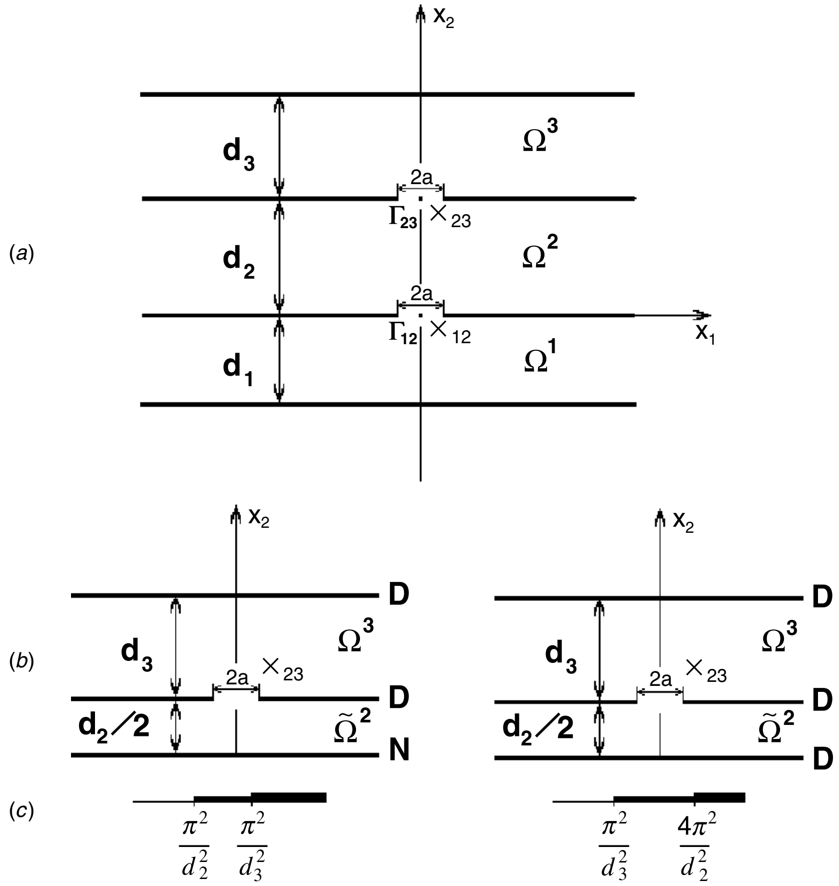

Figure 1. (*a*) Geometrical configuration of the system: $\Omega^1, \Omega^2, \Omega^3$ are the waveguides, and X_{12}, X_{23} are the centres of the coupling windows Γ_{12}, Γ_{23}, respectively. (*b*) Geometrical configuration for the reduced problems: $D(N)$ marks the type of boundary condition, Dirichlet (Neumann). (*c*) Branches of the continuous spectra of the operators for the reduced problems.

obtained only estimates and not asymptotics. Analogous estimates were obtained for the case of n coupling windows [8]. The asymptotics of the eigenvalue in question was obtained in [9, 10]. Some further results can be found in [11]. The method of matching of the asymptotic expansions (in a) for the corresponding solutions was used. The scheme of matching was a modification of that suggested in [12, 13]. The problem of resonance (quasi-eigenvalue) was considered in [14]. The asymptotics of resonances close to the Nth threshold is obtained for the problem of two coupled waveguides. It is very important for physical applications to investigate the behaviour of the resonance and the possibility of its transformation in the eigenvalue (see, for example, [15, 16]). The present paper is devoted to this problem.

We consider the spectral problem for the Dirichlet Laplacian $-\Delta$ in three two-dimensional strips Ω^1, Ω^2 and Ω^3 of widths d_1, d_2 and d_3, respectively, coupled through apertures of widths $2a$ (figure 1(*a*)). If $d_1 = d_3$, the main space $\mathcal{H}$ can be represented as an orthogonal sum, $\mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_s$, where $\mathcal{H}_a(\mathcal{H}_s)$ is a subspace of antisymmetric (symmetric) in respect to the centreline of Ω^2 functions. These subspaces are invariant subspaces of the Dirichlet Laplacian, consequently we have the corresponding representation of the operator: $-\Delta = -\Delta_a \oplus \Delta_s$. Hence, the problem, actually splits into two boundary problems, each for two coupled strips (figure 1(*b*)) (with the Dirichlet or Neumann boundary condition on the former centreline). The spectra of the corresponding operators $(-\Delta_a, -\Delta_s)$ are shown in figure 1(*c*). It is known (see above) that the operator $-\Delta_a$ has an eigenvalue close to the threshold. Consider the breaking of symmetry, namely, let $d_1 \neq d_3$ $(d_3 < d_1 < d_2 < 2d_3)$. In this case there is no reduction of the problem, and the mentioned eigenvalue becomes a resonance (quasi-eigenvalue). This effect is very important for the description of transport properties of the system. This is why it is interesting to look for this transition 'resonance-eigenvalue'. We deal with the asymptotics (in a) of the resonance. To look for the asymptotics of the resonance when $d_3 \to d_1$ we suppose that $d_1^2 = d_3^2 + \chi a^6$, $\chi > 0$. The main goal of the paper is to find the first terms of the asymptotic expansions of the resonance. We consider the Helmholtz equation $(\Delta + k^2)\varphi_a = 0$ with the Dirichlet boundary condition in a system of three coupled waveguides and we seek the quasi-eigenvalue $\lambda_a = k_a^2$ close to the threshold $\frac{\pi^2}{d_3^2}$. The method of matching of asymptotic expansions of the solutions of boundary value problems is used.

2. Resonance asymptotics

Let us introduce the local coordinate system $x_1^{(12)}, x_2^{(12)}$ $\left(x_1^{(23)}, x_2^{(23)}\right)$ with the origin at the centre of the opening Γ_{12} (Γ_{23}) in such a way that the axis $Ox_1^{(12)}$ $\left(Ox_1^{(23)}\right)$ coincides with the waveguide wall and $x_2^{(12)} > 0$ $\left(x_2^{(23)} > 0\right)$ in Ω^1 (Ω^2). We seek the asymptotic expansion of the quasi-eigenvalue in the following form:

$$\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} = \sum_{j=2}^{\infty} \sum_{i=0}^{[(j-1)/2]} k_{ji} a^j \left(\ln \frac{a}{a_0}\right)^i. \tag{2}$$

Here, a_0 is some characteristic length unit. We find some first coefficients k_{ji} in equation (2). The expansions of the corresponding quasi-eigenfunction in different domains are as follows:

$$\varphi_a(x) = -\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} \sum_{j=0}^{\infty} a^j P_{j+1}^{(1)}\left(D_y, \ln \frac{a}{a_0}\right) G_1(x, y, k)|_{y=X_{12}} \qquad x \in \Omega^1 \backslash S^{12}_{a_0(\frac{a}{a_0})^{1/2}} \tag{3}$$

$$\varphi_a(x) = \sum_{j=1}^{\infty} \sum_{i=0}^{[(j-1)/2]} v_{ji}^{(12)}\left(\frac{x}{a}\right) a^j \left(\ln \frac{a}{a_0}\right)^i \qquad x \in S^{12}_{2a_0(\frac{a}{a_0})^{1/2}} \tag{4}$$

$$\begin{aligned}\varphi_a(x) = \left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} \sum_{j=0}^{\infty} a^j &\left(P_{j+1}^{(21)}\left(D_y, \ln \frac{a}{a_0}\right) G_2(x, y, k)|_{y=X_{12}}\right. \\ &\left. + P_{j+1}^{(23)}\left(D_y, \ln \frac{a}{a_0}\right) G_2(x, y, k)|_{y=X_{23}}\right) \\ &x \in \Omega^2 \Big\backslash \left(S^{12}_{2a_0(\frac{a}{a_0})^{1/2}} \cup S^{23}_{2a_0(\frac{a}{a_0})^{1/2}}\right)\end{aligned} \tag{5}$$

$$\varphi_a(x) = \sum_{j=1}^{\infty} \sum_{i=0}^{[(j-1)/2]} v_{ji}^{(23)}\left(\frac{x}{a}\right) a^j \left(\ln \frac{a}{a_0}\right)^i \qquad x \in S^{23}_{2a_0(\frac{a}{a_0})^{1/2}} \tag{6}$$

$$\varphi_a(x) = -\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} \sum_{j=0}^{\infty} a^j P_{j+1}^{(3)}\left(D_y, \ln\frac{a}{a_0}\right) G_3(x, y, k)|_{y=X_{23}} \qquad x \in \Omega^3 \setminus S^{23}_{2a_0(\frac{a}{a_0})^{1/2}}. \tag{7}$$

Here X_{ij} is the centre of the opening Γ_{ij}, S_t^{ij} is the sphere of the radius t centred at X_{ij}, and $v_{ji} \in W^1_{2,\mathrm{loc}}(\Omega^1 \cup \Omega^2 \cup \Omega^3)$. $P_m^{(rs)}$ and $P_m^{(r)}$ are some polynomials on D_y (derivative operator) and $\ln \frac{a}{a_0}$, having the forms

$$\begin{aligned} P_1^{(r)}\left(D_y, \ln\frac{a}{a_0}\right) &= {}^r a_{10}^{(1)} \frac{\partial}{\partial n_y} P_m^{(r)}\left(D_y, \ln\frac{a}{a_0}\right) \\ &= \sum_{j=1}^{m-1} \sum_{i=0}^{[(j-1)/2]} {}^r a_{ji}^{(m)} \left(\ln\frac{a}{a_0}\right)^i D_y^{m-j+i} \qquad m = 2, 3, 4, \ldots \end{aligned} \tag{8}$$

$$\begin{aligned} P_1^{(rs)}\left(D_y, \ln\frac{a}{a_0}\right) &= {}^{rs} a_{10}^{(1)} \frac{\partial}{\partial n_y} P_m^{(rs)}\left(D_y, \ln\frac{a}{a_0}\right) \\ &= \sum_{j=1}^{m-1} \sum_{i=0}^{[(j-1)/2]} {}^{rs} a_{ji}^{(m)} \left(\ln\frac{a}{a_0}\right)^i D_y^{m-j+i} \qquad m = 2, 3, 4, \ldots \end{aligned} \tag{9}$$

where ${}^r a_{ji}^{(m)}, {}^{rs} a_{ji}^{(m)}$ are some constants, $D_y^{2j+1} = \frac{\partial^{2j+1}}{\partial n_y^{2j+1}}$, $D_y^{2j+2} = \frac{\partial^{2j+2}}{\partial l_y \partial n_y^{2j+1}}$; $l = (1, 0)$, $n = (0, -1)$. G_i is the Dirichlet Green function for the Helmholtz equation in the waveguide Ω^i:

$$\begin{aligned} G_s(x, y, k) = \sum_{n=1}^{\infty} \frac{1}{d_s} \left(\frac{n^2\pi^2}{d_s} - k^2\right)^{-1/2} \sin\frac{n\pi x_2}{d_s} \sin\frac{n\pi y_2}{d_s} \\ \times \exp\left(-\left(\frac{n^2\pi^2}{d_s} - k^2\right)^{1/2} |x_1 - y_1|\right). \end{aligned} \tag{10}$$

The derivative of the Green function can be represented in a neighbourhood of the boundary point for spectral parameter close to $\frac{\pi^2}{d_3^2}$ in the form:

$$\begin{aligned} D_y^j G_s(x, y, k)|_{y=X_{pq}} = \frac{1}{d_s} \sin\frac{\pi x_2}{d_s} D_x^j \left(\sin\frac{\pi x_2}{d_s}\right)\Bigg|_{x=X_{pq}} \left(\frac{\pi^2}{d_s^2} - k_a^2\right)^{-\frac{1}{2}} \\ + \Phi_j(x, k) \ln\frac{r}{a_0} + g_j^{(s)}(x, k) + \sum_{i=0}^{[j/2]} \sum_{t=0}^{j-2i-1} b_{it}^{(j)}(k) r^{-j+2(i+t)} \sin(j - 2i)\theta. \end{aligned} \tag{11}$$

Here (r, θ) are the polar coordinates. Functions $b_{it}^{(j)}(k)$, $\Phi_j(x, k)$, $g_j^{(l)}(x, k)$ are analytic in k in a neighbourhood of $\frac{\pi^2}{d_3}$; $\Phi_j(x, k) \in C^\infty(\Re^2)$ and are continuous in respect to x_2. It is known (see, for example, [13]) that $g_j^{(l)}(x, k) \in C^\infty(\Omega^l)$ and

$$b_{00}^{(j)} = \frac{(-1)^{[(j+1)/2]}(j-1)!}{\pi} \qquad b_{10}^{(3)} = \frac{k^2}{2\pi} \tag{12}$$

$$\Phi_{1n}(0, k) = -\frac{k^2}{2\pi}. \tag{13}$$

Functions $v_{ji}^{(mn)}(x/a)$ are solutions of boundary value problems which are obtained by the following way. We substitute series (2), (4) and (6) into the Helmholtz equation (for $k = k_a$). Then we replace the variables in the local coordinates $\xi = x/a$ and make equal terms of the same order of a and $\ln a$. Taking the formal limit $a \to 0$, we obtain the following boundary problem for $v_{ji}^{(mn)}(x/a)$

$$\Delta_\xi v_{ji}^{(mn)} = -\sum_{p=0}^{j-3}\sum_{q=0}^{[p/2]-1} \Lambda_{pq} v_{j-p-2,\,i-q}^{(mn)} \qquad \xi \in R^2\backslash\gamma,\ v_{ji}^{(mn)} = 0, \xi \in \bar{\gamma} \tag{14}$$

where $\bar{\gamma} = \{\xi : \xi_2 = 0, \xi_1 \in (-\infty, -1] \cup [1, \infty)\}$, and Λ_{pq} are the coefficients of the series

$$k_a^2 = \sum_p \sum_q \Lambda_{pq} a^p \left(\ln \frac{a}{a_0}\right)^q.$$

Let us define the operator $M_{pq}^{(ij)}$ on sums $U(x, a)$ of type (3), (5) and (7) in local coordinates with the centre X_{ij}. Replace $\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}}$ by equation (2) and terms with d_1 in equation (3) by the following manner. We expand coefficients of $U(x, a)$ in an asymptotic series for $r \to \infty$ and change the variables $\xi = \frac{x}{a}$ ($\ln r$ is replaced by $\ln \rho + \ln a$). We denote the sum of all terms of the type $a^p (\ln a)^q\, \mathrm{H}(\xi)$ by M_{pq}. Let $M_p^{(ij)} = \sum_q M_{pq}^{(ij)}$.

To match asymptotic expansions (3) and (4), (4) and (5), (5) and (6), (6) and (7), we should make equal the coefficients in the terms of the same order $a^p\left(\ln \frac{a}{a_0}\right)^q$. To find k_{20} it is sufficient to consider the terms of order a. Taking into account equations (11)–(13), we have

$$\lim_{k \to \frac{\pi}{d_3}} \left(-\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} P_1^{(1)} G_1(x^{(12)}, y, k)|_{y=X_{12}}\right) = \frac{\pi}{d_3^2}\,{}^1a_{10}^{(1)} \sin \frac{\pi x_2^{(12)}}{d_3}$$

$$\lim_{k \to \frac{\pi}{d_3}} \left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} \left(P_1^{(21)} G_2(x^{(12)}, y, k_a)|_{y=X_{12}} + P_1^{(23)} G_2(x^{(12)}, y, k_a)|_{y=X_{23}}\right) = 0$$

$$\lim_{k \to \frac{\pi}{d_3}} \left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} \left(P_1^{(21)} G_2(x^{(23)}, y, k_a)|_{y=X_{12}} + P_1^{(23)} G_2(x^{(23)}, y, k_a)|_{y=X_{23}}\right) = 0$$

$$\lim_{k \to \frac{\pi}{d_3}} \left(-\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} P_1^{(3)} G_3(x^{(23)}, y, k)|_{y=X_{23}}\right) = \frac{\pi}{d_3^2}\,{}^3a_{10}^{(1)} \sin \frac{\pi x_2^{(23)}}{d_3}.$$

Consequently,

$$a^{-1} M_1^{(12)} \left(-\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} P_1^{(1)} G_1(x^{(12)}, y, k_a)|_{y=X_{12}}\right)$$
$$= \frac{\pi^2}{d_3^3}\,{}^1a_{10}^{(1)} \rho_{12} \sin\theta_{12} + \frac{1}{\pi} k_{20}\,{}^1a_{10}^{(1)} \rho_{12}^{-1} \sin\theta_{12} \tag{15}$$

$$a^{-1} M_1^{(12)} \left(\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} \left(P_1^{(21)} G_2(x^{(12)}, y, k_a)|_{y=X_{12}} + P_1^{(23)} G_2(x^{(12)}, y, k_a)|_{y=X_{23}}\right)\right)$$
$$= -\frac{1}{\pi} k_{20}\,{}^{21}a_{10}^{(1)} \rho_{12}^{-1} \sin\theta_{12} \tag{16}$$

$$a^{-1} M_1^{(23)} \left(\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} P_1^{(21)} G_2(x^{(23)}, y, k_a)|_{y=X_{12}} + P_1^{(23)} G_2(x^{(23)}, y, k_a)|_{y=X_{23}}\right)$$
$$= -\frac{1}{\pi} k_{20}\,{}^{23}a_{10}^{(1)} \rho_{23}^{-1} \sin\theta_{23} \tag{17}$$

$$a^{-1}M_1^{(23)}\left(\left(\frac{\pi^2}{d_3^2}-k_a^2\right)^{\frac{1}{2}}P_1^{(3)}G_3(x^{(23)},y,k_a)|_{y=X_{23}}\right)$$
$$=\frac{\pi^2}{d_3^3}{}^3a_{10}^{(1)}\rho_{23}\sin\theta_{23}+\frac{1}{\pi}k_{20}{}^3a_{10}^{(1)}\rho_{23}^{-1}\sin\theta_{23}. \tag{18}$$

To match the terms (15) and (18) increasing at infinity, we choose the following solutions of equation (14)

$$v_{10}^{(12)}(\xi_{12})=\alpha_{12}Y_1(\xi_{12}^*) \tag{19}$$

$$v_{10}^{(23)}(\xi_{23})=\alpha_{23}Y_1(\xi_{23}) \tag{20}$$

where α_{12} and α_{23} are some constants. $Y_q(\xi_{mn})$ has the following asymptotics

$$Y_q(\xi_{mn})=\begin{cases}-\sum_{j=1}^{\infty}c_{qj}\rho_{mn}^{-j}\sin j\theta_{mn} & \xi_{mn,2}>0\\ \rho^q\sin q\theta+\sum_{j=1}^{\infty}c_{qj}\rho_{mn}^{-j}\sin j\theta_{mn} & \xi_{mn,2}<0.\end{cases} \tag{21}$$

The existence of such solutions is well known [17–19]. Here $\xi_{mn}=(\xi_{mn,1},\ \xi_{mn,2})$, $\xi_{mn}^*=(\xi_{mn,1},\ -\xi_{mn,2})$, c_{qj} is real, $c_{11}=1/4, c_{21}=2c_{12}=0, c_{13}=1/16, c_{31}=3/16$. Making equal coefficients of the terms $\rho_{12}\sin\theta_{12}$, $\rho_{12}^{-1}\sin\theta_{12}$ in equations (15) and (19), (19) and (16), and also coefficients of the terms $\rho_{23}\sin\theta_{23}$, $\rho_{23}^{-1}\sin\theta_{23}$ in equations (17) and (20), (20) and (18), we obtain the following homogeneous system of equations for the determination of ${}^1a_{10}^{(1)}, {}^{21}a_{10}^{(1)}, {}^{23}a_{10}^{(1)}, {}^3a_{10}^{(1)}, \alpha_{12}, \alpha_{23}$:

$$\begin{cases}\frac{\pi^2}{d_3^3}{}^1a_{10}^{(1)}=-\alpha_{12}\\ \frac{k_{20}}{\pi}{}^1a_{10}^{(1)}=-\frac{1}{4}\alpha_{12}\\ \frac{1}{4}\alpha_{12}=-\frac{1}{\pi}k_{20}{}^{21}a_{10}^{(1)}\\ -\frac{1}{\pi}k_{20}{}^{23}a_{10}^{(1)}=-\frac{1}{4}\alpha_{23}\\ \alpha_{23}=\frac{\pi^2}{d_3^3}{}^3a_{10}^{(1)}\\ \frac{1}{4}\alpha_{23}=\frac{1}{\pi}k_{20}{}^3a_{10}^{(1)}.\end{cases} \tag{22}$$

Excluding α_{12} and α_{23}, we reduce the system to the form

$$\begin{cases}\frac{\pi^2}{d_3^3}{}^1a_{10}^{(1)}-\frac{4k_{20}}{\pi}{}^{21}a_{10}^{(1)}=0\\ \frac{k_{20}}{\pi}{}^1a_{10}^{(1)}-\frac{k_{20}}{\pi}{}^{21}a_{10}^{(1)}=0\\ \frac{\pi^2}{d_3^3}{}^3a_{10}^{(1)}-\frac{4k_{20}}{\pi}{}^{23}a_{10}^{(1)}=0\\ \frac{k_{20}}{\pi}{}^3a_{10}^{(1)}-\frac{k_{20}}{\pi}{}^{23}a_{10}^{(1)}=0.\end{cases} \tag{23}$$

The condition of non-trivial solvability of the system gives us the equation for the determination of k_{20}:

$$\frac{k_{20}^2}{\pi^2}\left(\frac{4k_{20}}{\pi}-\frac{\pi^2}{d_3^3}\right)^2=0. \tag{24}$$

We choose such a solution of equation (24) for which all values of ${}^1a_{10}^{(1)}, {}^{21}a_{10}^{(1)}, {}^{23}a_{10}^{(1)}, {}^1a_{10}^{(3)}$ can be non-zero, namely,

$$k_{20}=\frac{\pi^3}{4d_3^3}. \tag{25}$$

For this value of k_{20} we obtain from equation (23) the following relations

$$ {}^{21}a_{10}^{(1)} = {}^{1}a_{10}^{(1)} \tag{26} $$

$$ {}^{23}a_{10}^{(1)} = {}^{3}a_{10}^{(1)}. \tag{27} $$

Furthermore, we have

$$ \lim_{k\to\frac{\pi}{d_-}} \left(-\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} P_3^{(1)} G_1(x^{(12)}, y, k)|_{y=X_{12}} \right) = -\frac{\pi^3}{d_3^4}\, {}^{1}a_{10}^{(3)} \sin\frac{\pi x_2^{(12)}}{d_3} $$

$$ \lim_{k\to\frac{\pi}{d_-}} \left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} \left(P_1^{(21)} G_2(x^{(12)}, y, k_a)|_{y=X_{12}} + P_1^{(23)} G_2(x^{(12)}, y, k_a)|_{y=X_{23}}\right) = 0 $$

$$ \lim_{k\to\frac{\pi}{d_-}} \left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} \left(P_1^{(21)} G_2(x^{(23)}, y, k_a)|_{y=X_{12}} + P_1^{(23)} G_2(x^{(23)}, y, k_a)|_{y=X_{23}}\right) = 0 $$

$$ \lim_{k\to\frac{\pi}{d_-}} \left(-\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} P_1^{(3)} G_3(x^{(23)}, y, k)|_{y=X_{23}} \right) = -\frac{\pi^3}{d_3^4}\, {}^{3}a_{10}^{(3)} \sin\frac{\pi x_2^{(23)}}{d_3}. $$

Hence,

$$ a^{-1} M_1^{(12)} \left(-\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} P_3^{(1)} G_1(x^{(12)}, y, k_a)|_{y=X_{12}} \right) = -\frac{2}{\pi} k_{20}\, {}^{1}a_{10}^{(3)} \rho_{12}^{-3} \sin 3\theta_{12} \tag{28} $$

$$ \begin{aligned} a^{-1} M_1^{(12)} &\left(-\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} \left(P_1^{(21)} G_2(x^{(12)}, y, k_a)|_{y=X_{12}} + P_1^{(23)} G_2(x^{(12)}, y, k_a)|_{y=X_{23}}\right) \right) \\ &= \frac{2}{\pi} k_{20}\, {}^{21}a_{10}^{(3)} \rho_{12}^{-3} \sin 3\theta_{12} \end{aligned} \tag{29} $$

$$ \begin{aligned} a^{-1} M_1^{(23)} &\left(-\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} \left(P_1^{(21)} G_2(x^{(23)}, y, k_a)|_{y=X_{12}} + P_1^{(23)} G_2(x^{(23)}, y, k_a)|_{y=X_{23}}\right) \right) \\ &= \frac{2}{\pi} k_{20}\, {}^{23}a_{10}^{(3)} \rho_{23}^{-3} \sin 3\theta_{23} \end{aligned} \tag{30} $$

$$ a^{-1} M_1^{(23)} \left(-\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} P_3^{(3)} G_2(x^{(23)}, y, k_a)|_{y=X_{23}} \right) = -\frac{2}{\pi} k_{20}\, {}^{3}a_{10}^{(3)} \rho_{23}^{-3} \sin 3\theta_{23}. \tag{31} $$

Making equal the coefficients of the terms $\rho_{12}^{-3}\sin\theta_{12}$ in equations (28) and (19), (19) and (29), and also the coefficients of the terms $\rho_{23}^{-3}\sin\theta_{23}$ in equations (30) and (20), (20) and (31), we come to the following system for the determination of ${}^{1}a_{10}^{(3)}, {}^{21}a_{10}^{(3)}, {}^{23}a_{10}^{(3)}, {}^{23}a_{10}^{(3)}, {}^{3}a_{10}^{(3)}, \alpha_{12}, \alpha_{23}$:

$$ \begin{cases} -\frac{2k_{20}}{\pi}\, {}^{1}a_{10}^{(3)} = -\frac{1}{16}\alpha_{12} \\ \frac{1}{16}\alpha_{12} = \frac{2k_{20}}{\pi}\, {}^{21}a_{10}^{(3)} \\ \frac{2k_{20}}{\pi}\, {}^{23}a_{10}^{(3)} = -\frac{1}{16}\alpha_{23} \\ \frac{1}{16}\alpha_{23} = -\frac{2k_{20}}{\pi}\, {}^{3}a_{10}^{(3)}. \end{cases} \tag{32} $$

Replacing α_{12} by $-\frac{4k_{20}}{\pi}{}^1a_{10}^{(1)}$ and α_{23} by $\frac{4k_{20}}{\pi}{}^3a_{10}^{(1)}$ in accordance with the second and the last equations of the system (22) and taking into account that $k_{20} \neq 0$, we can obtain the relations between ${}^1a_{10}^{(3)}$, ${}^{21}a_{10}^{(3)}$, ${}^{23}a_{10}^{(3)}$, ${}^{23}a_{10}^{(3)}$, ${}^3a_{10}^{(3)}$ and ${}^1a_{10}^{(1)}$, ${}^3a_{10}^{(1)}$:

$$ {}^{21}a_{10}^{(3)} = {}^1a_{10}^{(3)} = -\frac{1}{8}{}^1a_{10}^{(1)} \tag{33} $$

$$ {}^{23}a_{10}^{(3)} = {}^3a_{10}^{(3)} = -\frac{1}{8}{}^3a_{10}^{(1)}. \tag{34} $$

The value of k_{40} is determined by matching the terms of order a^3 in equations (3) and (4), (4) and (5), (5) and (6), (6) and (7). Firstly, we expand functions $g_1^{(1)}(x^{(12)}, k)$, $g_1^{(2)}(x^{(12)}, k)$, $\Phi_1(x^{(12)}, k)$ in a series in powers of $x_1^{(12)}, x_2^{(12)}$, and functions $g_1^{(2)}(x^{(23)}, k)$, $g_1^{(3)}(x^{(23)}, k)$, $\Phi_1(x^{(23)}, k)$ in a series in powers of $x_1^{(23)}, x_2^{(23)}$ in the neighbourhood of X_{12}. Taking into account that function $\Phi_j(x^{(12)}, k)$ is antisymmetric with respect to $x_2^{(12)}$, we obtain

$$ \Phi_j(0,k) = 0 \qquad \left.\frac{\partial^i \Phi_j(x^{(12)}, k)}{\left(\partial x_1^{(12)}\right)^i}\right|_{x^{(12)}=(0,0)} = 0. $$

We also have

$$ G_1(x^{(12)}, y, k)|_{x_2^{(12)}=0} \equiv 0 $$

and

$$ D_y^j G_1(x^{(12)}, y, k)|_{y=X_{12}, x_2^{(12)}=0} \equiv 0. $$

We obtain from equation (11)

$$ \begin{aligned} g_j^{(1)}(x^{(12)}, k)|_{x_2^{(12)}=0} &= D_y^j G_1(x^{(12)}, y, k)|_{y=X_{12}, x_2^{(12)}=0} - \frac{2}{d_1} \sin \left.\frac{\pi x_2^{(12)}}{d_1}\right|_{x_2^{(12)}=0} \\ &\quad \times D_x^j \left.\left(\sin \frac{\pi x_2^{(12)}}{d_1}\right)\right|_{x_2^{(12)}=0} \left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{-\frac{1}{2}} - \Phi_1(x^{(12)}, k)|_{x_2^{(12)}=0} \ln \frac{r_{12}}{a_0} \\ &\quad - \sum_{i=0}^{[j/2]} \sum_{t=0}^{j-2i-1} b_{it}^{(j)}(k) r_{12}^{-j+2(i+t)} \sin(j-2i)\theta_{12}|_{\theta_{12}=0} \equiv 0, \end{aligned} $$

and hence,

$$ \left.\frac{\partial g_j^{(1)}(x^{(12)}, k)}{\partial x_1^{(12)}}\right|_{x^{(12)}=(0,0)} = 0. $$

We introduce the following notation

$$ g_x^{(1)} = \left.\frac{\partial g_1^{(1)}(x^{(12)}, k)}{\partial x_2^{(12)}}\right|_{x^{(12)}=(0,0), k=k_0} $$

where $k_0 = \frac{\pi^2}{d_3^2}$. We obtain from equation (13)

$$ \left.\frac{\partial^i \Phi_j(x^{(12)}, k)}{\left(\partial x_2^{(12)}\right)^i}\right|_{x^{(12)}=(0,0)} = \frac{k^2}{2\pi}. $$

Let us consider also the functions $\Phi_j(x^{(23)}, k)$, $g_j^{(2)}$, $g_j^{(3)}$, G_2, G_3 in an analogous way. Then we replace the function $\sin \frac{\pi x_2^{(12)}}{d_1}$ by a series in powers of $x_2^{(12)}$, we make the transition to new variables ρ_{12}, θ_{12} and we use the identity $\rho_{12}^3 \sin^3\theta_{12} = \rho_{12}^3(3\sin\theta_{12} - \sin 3\theta_{12})/4$. The analogous transformation is made with the function $\sin\frac{\pi x_2^{(23)}}{d_1}$.

Let us construct the asymptotics of $P_1^{(21)}G_2(x,y,k)|_{y=X_{12}}$ in the neighbourhood of X_{23} and $P_1^{(23)}G_2(x,y,k)|_{y=X_{23}}$ in the neighbourhood of X_{12}. From equation (10) we have

$$\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} P_1^{(21)}G_2(x^{(12)}, y, k_a)|_{y=X_{12}} = -\frac{\pi^{21}a_{10}^{(1)}}{d_2 d_3}\sqrt{\frac{\pi^2 - k_a^2 d_3^2}{\pi^2 - k_a^2 d_2^2}} \sin\frac{\pi x_2^{(12)}}{d_2}$$
$$- \frac{\pi}{d_2^2}\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} \sum_{n=2}^{\infty} \sin\frac{n\pi x_2^{(12)}}{d_2} \exp\left(-\left(\frac{n^2\pi^2}{d_l} - k_a^2\right)^{\frac{1}{2}} |x_1|\right). \tag{35}$$

Note that only the first term on the right-hand side is imaginary, and others are real.

$$a^{-3}M_3^{(12)}\left(-\left(\frac{\pi^2}{d_3^2} - k_a^2\right)^{\frac{1}{2}} \left(P_1^{(1)} + P_3^{(1)}\right)G_1(x^{(12)}, y, k_a)|_{y=X_{12}}\right)$$
$$= \frac{\pi^4}{24 d_3^5}{}^1a_{10}^{(1)}\rho_{12}^3 \sin 3\theta_{12} - \frac{\pi^4}{8 d_3^5}{}^1a_{10}^{(1)}\rho_{12}^3 \sin\theta_{12} - \frac{\pi}{2 d_3^2}{}^1a_{10}^{(1)}k_{20}\rho_{12}\ln\rho_{12}\sin\theta_{12}$$
$$+ \left(\frac{\pi^4\chi}{2k_{20}^2 d_3^7}{}^1a_{10}^{(1)} - {}^1a_{10}^{(1)}g_x^{(1)} - \frac{\pi^4}{d_3^5}{}^1a_{10}^{(3)}\right)\rho_{12}\sin\theta_{12}$$
$$+ \left(\frac{1}{\pi}k_{40}{}^1a_{10}^{(1)} - \frac{\pi}{2d_3^2}k_{20}{}^1a_{10}^{(3)}\right)\rho_{12}^{-1}\sin\theta_{12} - k_{20}{}^1a_{10}^{(3)}b_{01}^{(3)}(k_0)\rho_{12}^{-1}\sin 3\theta_{12} \tag{36}$$

$$a^{-3}M_{30}^{(12)}\left(\left(\frac{\pi^2}{d_2^2} - k_a^2\right)^{\frac{1}{2}} \left(\left(P_1^{(21)} + P_3^{(21)}\right)G_2(x^{(12)}, y, k_a)|_{y=X_{12}}\right.\right.$$
$$\left.\left. + \left(P_1^{(23)} + P_3^{(23)}\right)G_2(x^{(12)}, y, k_a)|_{y=X_{23}}\right)\right)$$
$$= \left(\left(g_x^2 - \frac{\pi d_3}{d_2^2\sqrt{d_3^2 - d_2^2}}\right)k_{20}{}^{21}a_{10}^{(1)} + k_{20}f^{23}a_{10}^{(1)}\right)\rho_{12}\sin\theta_{12}$$
$$+ \frac{\pi}{2d_3^2}{}^{21}a_{10}^{(1)}k_{20}\rho_{12}\ln\rho_{12}\sin\theta_{12} - \left(\frac{1}{\pi}k_{40}{}^{21}a_{10}^{(1)} - \frac{\pi}{2d_3^2}k_{20}{}^{21}a_{10}^{(3)}\right)\rho_{12}^{-1}\sin\theta_{12}$$
$$+ k_{20}{}^{21}a_{10}^{(3)}b_{01}^{(3)}(k_0)\rho_{12}^{-1}\sin 3\theta_{12} \tag{37}$$

$$a^{-3}M_{30}^{(23)}\left(\left(\frac{\pi^2}{d_2^2} - k_a^2\right)^{\frac{1}{2}} \left(\left(P_1^{(21)} + P_3^{(21)}\right)G_2(x^{(23)}, y, k_a)|_{y=X_{12}}\right.\right.$$
$$\left.\left. + \left(P_1^{(23)} + P_3^{(23)}\right)G_2(x^{(23)}, y, k_a)|_{y=X_{23}}\right)\right)$$
$$= \left(\left(g_x^2 - \frac{\pi d_3}{d_2^2\sqrt{d_3^2 - d_2^2}}\right)k_{20}{}^{23}a_{10}^{(1)} + k_{20}f^{21}a_{10}^{(1)}\right)\rho_{23}\sin\theta_{23}$$
$$+ \frac{\pi}{2d_3^2}{}^{23}a_{10}^{(1)}k_{20}\rho_{23}\ln\rho_{23}\sin\theta_{23} - \left(\frac{1}{\pi}k_{40}{}^{23}a_{10}^{(1)} - \frac{\pi}{2d_3^2}k_{20}{}^{23}a_{10}^{(3)}\right)\rho_{23}^{-1}\sin\theta_{23}$$
$$+ k_{20}{}^{23}a_{10}^{(3)}b_{01}^{(3)}(k_0)\rho_{23}^{-1}\sin 3\theta_{23} \tag{38}$$

$$
\begin{aligned}
& a^{-3}M_{30}^{(23)}\left(-\left(\frac{\pi^2}{d_3^2}-k_a^2\right)^{\frac{1}{2}}\left(P_1^{(1)}+P_3^{(1)}\right)G_3(x^{(23)},y,k_a)|_{y=X_{23}}\right)\\
&\quad=\frac{\pi^4}{24d_3^5}{}^3a_{10}^{(1)}\rho_{23}^3\sin 3\theta_{23}-\frac{\pi^4}{8d_3^5}{}^3a_{10}^{(1)}\rho_{23}^3\sin\theta_{23}-\frac{\pi}{2d_3^2}{}^3a_{10}^{(1)}k_{20}\rho_{23}\ln\rho_{23}\sin\theta_{23}\\
&\quad+\left(-{}^3a_{10}^{(1)}g_x^{(1)}-\frac{\pi^4}{d_3^5}{}^3a_{10}^{(3)}\right)\rho_{23}\sin\theta_{23}\\
&\quad+\left(\frac{1}{\pi}k_{40}{}^3a_{10}^{(1)}-\frac{\pi}{2d_3^2}k_{20}{}^3a_{10}^{(3)}\right)\rho_{23}^{-1}\sin\theta_{23}-k_{20}{}^3a_{10}^{(3)}b_{01}^{(3)}(k_0)\rho_{23}^{-1}\sin 3\theta_{23}. \qquad (39)
\end{aligned}
$$

Let us construct the asymptotics of $v_{30}^{(12)}(\xi_{12})$ and $v_{30}^{(23)}(\xi_{23})$. In accordance with equation (14), we come to the following boundary problem for $v_{30}^{(12)}(\xi_{12})$:

$$
\begin{aligned}
&\Delta_{\xi_{12}}v_{30}^{(12)}(\xi_{12})=-k_0^2v_{10}(\xi_{12}) &&\xi_{12}\in R^2\backslash\bar{\gamma}_{12}\\
&v_{30}(\xi_{12})=0 &&\xi_{12}\in\bar{\gamma}_{12}
\end{aligned}
\qquad (40)
$$

where $\bar{\gamma}_{12}=\{\xi_{12}:\xi_{12,2}=0,\xi_{12,1}\in(-\infty,-1]\cup[1,\infty)\}$. Let $\tilde{v}_{30}^{(12)}(\xi_{12})$ be a particular solution of the inhomogeneous Laplace equation (Poisson equation) which satisfies the boundary condition, and let $\hat{v}_{30}^{(12)}(\xi_{12})$ $\left(\hat{v}_{30}^{(12)}(\xi_{12})=v_{30}^{(12)}(\xi_{12})-\tilde{v}_{30}^{(12)}(\xi_{12})\right)$ be the solution of the corresponding homogeneous equation, satisfying the boundary condition. We can easily show (by substitution into equation (40)) that the asymptotics of $\tilde{v}_{30}^{(12)}(\xi_{12})$ is as follows:

$$
\tilde{v}_{30}^{(12)}(\xi_{12})=\frac{\pi^2}{d_3^3}{}^1a_{10}^{(1)}\begin{cases}-\frac{1}{8}\rho_{12}^3\sin\theta_{12}-\frac{c_{11}}{2}\rho_{12}\ln\rho_{12}\sin\theta_{12}\\ \quad+\sum_{j=3}^{\infty}\frac{c_{1j}\rho_{12}^{2-j}\sin j\theta_{12}}{4(j-1)} & \xi_{12,2}>0\\ +\frac{c_{11}}{2}\rho_{12}\ln\rho_{12}\sin\theta_{12}-\sum_{j=3}^{\infty}\frac{c_{1j}\rho_{12}^{2-j}\sin j\theta_{12}}{4(j-1)} & \xi_{12,2}<0.\end{cases}\qquad (41)
$$

To match the terms of (36) and (37) increasing at infinity, we choose the following form of $\hat{v}_{30}^{(12)}(\xi_{12})$

$$
\hat{v}_{30}^{(12)}(\xi_{12})=\beta_{12}Y_1(\xi_{12})+\gamma_{12}Y_1(\xi_{12}^*)+\delta_{12}Y_3(\xi_{12}^*) \qquad (42)
$$

where $\beta_{12},\gamma_{12},\delta_{12}$ are constants. Finally, the asymptotics of $v_{30}^{(12)}(\xi_{12})$ has the following form:

$$
\begin{aligned}
v_{30}^{(12)}(\xi_{12})&=-\delta_{12}\rho_{12}^3\sin 3\theta_{12}-\gamma_{12}\rho_{12}\sin\theta_{12}-\frac{\pi^2}{8d_3^3}{}^1a_{10}^{(1)}\rho_{12}^3\sin\theta_{12}\\
&\quad-\frac{\pi^2c_{11}{}^1a_{10}^{(1)}}{2d_3^3}\rho_{12}\ln\rho_{12}\sin\theta_{12}+\frac{\pi^{2\,1}a_{10}^{(1)}}{d_3^3}\sum_{j=3}^{\infty}\frac{c_{1j}\rho_{12}^{2-j}\sin j\theta_{12}}{4(j-1)}\\
&\quad-\sum_{j=1}^{\infty}((\beta_{12}+\gamma_{12})c_{1j}+\delta_{12}c_{3j})\rho_{12}^{-j}\sin j\theta_{12} \qquad \xi_{12,2}>0 \qquad (43)\\
v_{30}^{(12)}(\xi_{12})&=\beta_{12}\rho_{12}\sin\theta_{12}+\frac{\pi^2c_{11}{}^1a_{10}^{(1)}}{2d_3^3}\rho_{12}\ln\rho_{12}\sin\theta_{12}-\frac{\pi^{2\,1}a_{10}^{(1)}}{d_3^3}\sum_{j=3}^{\infty}\frac{c_{1j}\rho_{12}^{2-j}\sin j\theta_{12}}{4(j-1)}\\
&\quad+\sum_{j=1}^{\infty}((\beta_{12}+\gamma_{12})c_{1j}+\delta_{12}c_{3j})\rho_{12}^{-j}\sin j\theta_{12} \qquad \xi_{12,2}<0.
\end{aligned}
$$

We construct the asymptotics of $v_{30}^{(23)}(\xi_{23})$ analogously and obtain

$$v_{30}^{(23)}(\xi_{23}) = -\gamma_{23}\rho_{23}\sin\theta_{23} + \frac{\pi^2 c_{11}{}^3a_{10}^{(1)}}{2d_3^3}\rho_{23}\ln\rho_{23}\sin\theta_{23} - \frac{\pi^{23}a_{10}^{(1)}}{d_3^3}\sum_{j=3}^{\infty}\frac{c_{1j}\rho_{23}^{2-j}\sin j\theta_{23}}{4(j-1)}$$
$$-\sum_{j=1}^{\infty}((\beta_{23}+\gamma_{23})c_{1j}+\delta_{23}c_{3j})\rho_{23}^{-j}\sin j\theta_{23} \qquad \xi_{23,2}>0$$
$$v_{30}^{(23)}(\xi_{23}) = \delta_{23}\rho_{23}^3\sin 3\theta_{23} + \beta_{23}\rho_{23}\sin\theta_{23} - \frac{\pi^2}{8d_3^3}{}^3a_{10}^{(1)}\rho_{23}^3\sin\theta_{23} \qquad (44)$$
$$-\frac{\pi^2 c_{11}{}^3a_{10}^{(1)}}{2d_3^3}\rho_{23}\ln\rho_{23}\sin\theta_{23} + \frac{\pi^{23}a_{10}^{(1)}}{d_3^3}\sum_{j=3}^{\infty}\frac{c_{1j}\rho_{23}^{2-j}\sin j\theta_{23}}{4(j-1)}$$
$$+\sum_{j=1}^{\infty}((\beta_{23}+\gamma_{23})c_{1j}+\delta_{23}c_{3j})\rho_{23}^{-j}\sin j\theta_{23} \qquad \xi_{23,2}<0.$$

Making equal the coefficients of the terms $\rho_{12}^3\sin 3\theta_{12}$, $\rho_{12}\sin\theta_{12}$, $\rho_{12}^{-1}\sin\theta_{12}$ in equations (36) and (43), (43) and (37), and also coefficients of the terms $\rho_{23}^3\sin 3\theta_{23}$, $\rho_{23}\sin\theta_{23}$, $\rho_{23}^{-1}\sin\theta_{23}$ in equations (38) and (44), (44) and (39), we obtain the following linear system for the determination of ${}^1a_{10}^{(3)}$, ${}^{21}a_{10}^{(3)}$, ${}^{23}a_{10}^{(3)}$, ${}^{23}a_{10}^{(3)}$, ${}^3a_{10}^{(3)}$, β_{12}, γ_{12}, δ_{12}, β_{23}, γ_{23}, δ_{23}:

$$\begin{cases}
\frac{\pi^4}{24d_3^5}{}^1a_{10}^{(1)} = -\delta_{12}\\
\frac{\pi^4\chi}{2k_{20}^2d_3^7}{}^1a_{10}^{(1)} - {}^1a_{10}^{(1)}g_x^{(1)} - \frac{\pi^4}{d_3^5}{}^1a_{10}^{(3)} = -\gamma_{12}\\
\frac{1}{\pi}k_{40}{}^1a_{10}^{(1)} - \frac{\pi}{2d_3^2}k_{20}{}^1a_{10}^{(3)} = -\frac{\beta_{12}+\gamma_{12}}{4} - \frac{3\delta_{12}}{16}\\
\beta_{12} = \left(g_x^2 - \frac{\pi d_3}{d_2^2\sqrt{d_3^2-d_2^2}}\right)k_{20}{}^{21}a_{10}^{(1)} + k_{20}f^{23}a_{10}^{(1)}\\
\frac{\beta_{12}+\gamma_{12}}{4} + \frac{3\delta_{12}}{16} = -\frac{1}{\pi}k_{40}{}^{21}a_{10}^{(1)} + \frac{\pi}{2d_3^2}k_{20}{}^{21}a_{10}^{(3)}\\
\left(g_x^2 - \frac{\pi d_3}{d_2^2\sqrt{d_3^2-d_2^2}}\right)k_{20}{}^{23}a_{10}^{(1)} + k_{20}f^{21}a_{10}^{(1)} = -\gamma_{23}\\
-\frac{1}{\pi}k_{40}{}^{23}a_{10}^{(1)} + \frac{\pi}{2d_3^2}k_{20}{}^{23}a_{10}^{(3)} = -\frac{\beta_{23}+\gamma_{23}}{4} - \frac{\delta_{23}}{4}\\
\delta_{23} = \frac{\pi^4}{24d_3^5}{}^3a_{10}^{(1)}\\
\beta_{23} = -{}^3a_{10}^{(1)}g_x^{(1)} - \frac{\pi^4}{d_3^5}{}^3a_{10}^{(3)}\\
\frac{\beta_{23}+\gamma_{23}}{4} + \frac{3\delta_{23}}{16} = \frac{1}{\pi}k_{40}{}^3a_{10}^{(1)} - \frac{\pi}{2d_3^2}k_{20}{}^3a_{10}^{(3)}.
\end{cases} \qquad (45)$$

Excluding β_{12}, γ_{12}, δ_{12}, β_{23}, γ_{23}, δ_{23}, ${}^{21}a_{10}^{(3)}$, ${}^{23}a_{10}^{(3)}$, replacing ${}^1a_{10}^{(3)}$ by $-\frac{1}{8}{}^1a_{10}^{(1)}$, and ${}^3a_{10}^{(3)}$ by $-\frac{1}{8}{}^3a_{10}^{(1)}$ (in accordance with equations (33) and (34) and substituting the value of k_{20} from equation (25), we come to the following system of two equations for the determination of ${}^1a_{10}^{(1)}$ and ${}^3a_{10}^{(1)}$:

$$\begin{cases}
\left(\frac{\pi^3}{4d_3^3}\left(g_x^1+g_x^2\right) - \frac{\pi^4}{4d_3^2d_2^2\sqrt{d_3^2-d_2^2}} + \frac{4k_{40}}{\pi} - \frac{3\pi^4}{32d_3^5} - \frac{8\chi}{\pi^2d_3}\right){}^1a_{10}^{(1)} + \frac{1}{4}k_{20}f^3a_{10}^{(1)} = 0\\
\frac{1}{4}k_{20}f^1a_{10}^{(1)} + \left(\frac{\pi^3}{4d_3^3}\left(g_x^3+g_x^2\right) - \frac{\pi^4}{4d_3^2d_2^2\sqrt{d_3^2-d_2^2}} + \frac{4k_{40}}{\pi} - \frac{3\pi^4}{32d_3^5}\right){}^3a_{10}^{(1)} = 0.
\end{cases} \qquad (46)$$

The condition of non-trivial solvability of equation (46) gives us the equation for the determination of k_{40}:

$$\left(\frac{\pi^3}{4d_3^3}(g_x^1+g_x^2)-\frac{\pi^4}{4d_3^2d_2^2\sqrt{d_3^2-d_2^2}}+\frac{4k_{40}}{\pi}-\frac{3\pi^4}{32d_3^5}-\frac{8\chi}{\pi^2d_3}\right)$$
$$\times\left(\frac{\pi^3}{4d_3^3}\left(g_x^3+g_x^2\right)-\frac{\pi^4}{4d_3^2d_2^2\sqrt{d_3^2-d_2^2}}+\frac{4k_{40}}{\pi}-\frac{3\pi^4}{32d_3^5}\right)=\frac{\pi^6}{16d_3^6}f^2. \tag{47}$$

Consequently, we obtain the following value of k_{40}

$$k_{40}=-\frac{\pi^4}{32d_3^3}\left(g_x^{(1)}+g_x^{(2)}\right)+\frac{\pi^5}{16d_3^2d_2^2\sqrt{d_3^2-d_2^2}}+\frac{3\pi^5}{128d_3^5}-\frac{\chi}{\pi d_3}$$
$$+\frac{1}{2}\left(3\left(\frac{\pi^4}{16d_3^3}g_x^{(2)}-\frac{\pi^5}{16d_3^2d_2^2\sqrt{d_3^2-d_2^2}}-\frac{3\pi^5}{64d_3^5}\right)^2\right.$$
$$\left.+\left(\frac{\pi^4}{16d_3^3}\left(g_x^{(1)}+g_x^{(2)}\right)+\frac{2\chi}{\pi d_3}\right)^2+\frac{\pi^8f^2}{256d_3^6}\right)^{\frac{1}{2}}. \tag{48}$$

For $\chi=0$ we obtain

$$k_{40}=\frac{\pi^4}{16d_3^3}\left(\frac{3\pi}{8d_3^2}-\left(g_x^{(13)}+g_x^2\right)+\mathrm{Re}\,f\right) \tag{49}$$

where $g_x^{(13)}=g_x^{(1)}=g_x^{(3)}$.

To find k_{41} we match terms of order $a^3\ln\frac{a}{a_0}$, and we have

$$a^{-3}\left(\ln\frac{a}{a_0}\right)^{-1}M_{31}^{(12)}\left(-\left(\frac{\pi^2}{d_3^2}-k_a^2\right)^{\frac{1}{2}}P_1^{(1)}G_1(x^{(12)},y,k_a)|_{y=X_{12}}\right)$$
$$=-\frac{\pi}{2d_3^2}{}^1a_{10}^{(1)}k_{20}\rho_{12}\sin\theta_{12}+\frac{1}{\pi}{}^1a_{10}^{(1)}k_{41}\rho_{12}^{-1}\sin\theta_{12} \tag{50}$$

$$a^{-3}\left(\ln\frac{a}{a_0}\right)^{-1}\left(M_{31}^{(12)}\left(\frac{\pi^2}{d_3^2}-k_a^2\right)^{\frac{1}{2}}\left(P_1^{(21)}G_2(x^{(12)},y,k_a)|_{y=X_{12}}\right.\right.$$
$$\left.\left.+P_1^{(23)}G_2(x^{(12)},y,k_a)|_{y=X_{23}}\right)\right)$$
$$=\frac{\pi}{2d_3^2}{}^{21}a_{10}^{(1)}k_{20}\rho_{12}\sin\theta_{12}-\frac{1}{\pi}{}^{21}a_{10}^{(1)}k_{41}\rho_{12}^{-1}\sin\theta_{12} \tag{51}$$

$$a^{-3}\left(\ln\frac{a}{a_0}\right)^{-1}M_{31}^{(23)}\left(\left(\frac{\pi^2}{d_3^2}-k_a^2\right)^{\frac{1}{2}}\left(P_1^{(21)}G_2(x^{(23)},y,k_a)|_{y=X_{12}}\right.\right.$$
$$\left.\left.+P_1^{(23)}G_2(x^{(23)},y,k_a)|_{y=X_{23}}\right)\right)$$
$$=\frac{\pi}{2d_3^2}{}^{23}a_{10}^{(1)}k_{20}\rho_{23}\sin\theta_{23}-\frac{1}{\pi}{}^{23}a_{10}^{(1)}k_{41}\rho_{23}^{-1}\sin\theta_{23} \tag{52}$$

$$a^{-3}\left(\ln\frac{a}{a_0}\right)^{-1}M_{31}^{(23)}\left(-\left(\frac{\pi^2}{d_3^2}-k_a^2\right)^{\frac{1}{2}}P_1^{(3)}G_3(x^{(23)},y,k_a)|_{y=X_{23}}\right)$$
$$=-\frac{\pi}{2d_3^2}{}^3a_{10}^{(1)}k_{20}\rho_{23}\sin\theta_{23}+\frac{1}{\pi}{}^3a_{10}^{(1)}k_{41}\rho_{23}^{-1}\sin\theta_{23}. \quad (53)$$

To match the terms in equations (50)–(53) increasing at infinity, we choose $v_{31}^{(12)}(\xi_{12})$ and $v_{31}^{(12)}(\xi_{23})$ as follows:

$$v_{31}^{(12)}(\xi_{12})=\eta_{12}Y_1(\xi_{12})+\mu_{12}Y_1(\xi_{12}^*) \quad (54)$$

$$v_{31}^{(23)}(\xi_{23})=\eta_{23}Y_1(\xi_{23})+\mu_{23}Y_1(\xi_{23}^*). \quad (55)$$

Making equal the terms of order $\rho_{12}\sin\theta_{12},\rho_{12}^{-1}\sin\theta_{12}$ in equations (50) and (54), (54) and (51) and also the terms of order $\rho_{23}\sin\theta_{23},\rho_{23}^{-1}\sin\theta_{23}$ in equations (52) and (55), (55) and (53), we come to the following linear homogeneous system for ${}^1a_{10}^{(1)},{}^{21}a_{10}^{(1)},{}^{23}a_{10}^{(1)},{}^3a_{10}^{(1)},\eta_{12},\mu_{12},\eta_{23},\mu_{23}$:

$$\begin{cases}-\frac{\pi}{2d_3^2}{}^1a_{10}^{(1)}k_{20}=-\mu_{12}\\ \frac{1}{\pi}{}^1a_{10}^{(1)}k_{41}=-\frac{\mu_{12}+\eta_{12}}{4}\\ \eta_{12}=\frac{\pi}{2d_3^2}{}^{21}a_{10}^{(1)}k_{20}\\ \frac{\mu_{12}+\eta_{12}}{4}=-\frac{1}{\pi}{}^{21}a_{10}^{(1)}k_{41}\\ \frac{\pi}{2d_3^2}{}^{23}a_{10}^{(1)}k_{20}=-\mu_{23}\\ -\frac{1}{\pi}{}^{23}a_{10}^{(1)}k_{41}=-\frac{\mu_{23}+\eta_{23}}{4}\\ \eta_{23}=-\frac{\pi}{2d_3^2}{}^3a_{10}^{(1)}k_{20}\\ \frac{\mu_{12}+\eta_{12}}{4}=\frac{1}{\pi}{}^3a_{10}^{(1)}k_{41}.\end{cases} \quad (56)$$

Excluding $\eta_{12},\mu_{12},\eta_{23},\mu_{23}$, we obtain the system of four equations:

$$\begin{cases}\left(\frac{\pi k_{20}}{2d_3^2}+\frac{4k_{41}}{\pi}\right){}^1a_{10}^1+\frac{\pi k_{20}}{2d_3^2}{}^{21}a_{10}^{(1)}=0\\ k_{41}{}^1a_{10}^{(1)}-k_{41}{}^{21}a_{10}^{(1)}=0\\ \left(\frac{\pi k_{20}}{2d_3^2}+\frac{4k_{41}}{\pi}\right){}^3a_{10}^1+\frac{\pi k_{20}}{2d_3^2}{}^{23}a_{10}^{(1)}=0\\ k_{41}{}^3a_{10}^{(1)}-k_{41}{}^{23}a_{10}^{(1)}=0.\end{cases} \quad (57)$$

The condition of non-trivial solvability of equation (57) gives us the equation for the determination of k_{41}:

$$k_{41}^2\left(\frac{\pi k_{20}}{d_3^2}+\frac{4k_{41}}{\pi}\right)^2=0. \quad (58)$$

Choosing a non-zero solution, we obtain

$$k_{41}=-\frac{\pi^2}{4d_3^2}k_{20}=-\frac{\pi^5}{16d_3^5}. \quad (59)$$

Let us summarize the results. The first terms of the asymptotic expansion of the resonance are as follows:

$$k_a^2=k_0^2-k_{20}^2a^4-2k_{20}\left(k_{40}+k_{41}\ln\frac{a}{a_0}\right)a^6-\left(k_{40}^2+2k_{40}k_{41}\ln\frac{a}{a_0}+k_{41}^2\left(\ln\frac{a}{a_0}\right)^2\right)a^8$$
$$+\text{higher-order terms} \quad (60)$$

where

$$
\begin{aligned}
k_0^2 &= \frac{\pi^2}{d_3^2} \\
k_{20} &= \frac{\pi^3}{4d_3^3} \\
k_{40} &= -\frac{\pi^4}{32d_3^3}\left(g_x^{(1)} + g_x^{(2)}\right) + \frac{\pi^5}{16d_3^2 d_2^2\sqrt{d_3^2 - d_2^2}} + \frac{3\pi^5}{128d_3^5} - \frac{\chi}{\pi d_3} \\
&\quad + \frac{1}{2}\left(3\left(\frac{\pi^4}{16d_3^3}g_x^{(2)} - \frac{\pi^5}{16d_3^2 d_2^2\sqrt{d_3^2 - d_2^2}} - \frac{3\pi^5}{64d_3^5}\right)^2\right. \\
&\quad \left. + \left(\frac{\pi^4}{16d_3^3}\left(g_x^{(1)} + g_x^{(2)}\right) + \frac{2\chi}{\pi d_3}\right)^2 + \frac{\pi^8 f^2}{256d_3^6}\right)^{\frac{1}{2}} \\
k_{41} &= -\frac{\pi^5}{16d_3^5}.
\end{aligned}
\tag{61}
$$

We can see that, if $\chi = 0$, then k_{40} becomes real (equation (49)). This corresponds to the fact that in this case we have an eigenvalue instead of the resonance.

3. Discussion

The existence of eigenvalues, due to obstacles placed symmetrically in between parallel walls having either Neumann or Dirichlet conditions imposed upon them, are well known to occur for frequencies below the continuous spectrum or channel cut-off (threshold) and for a range of geometrical configurations. These eigenvalues are stable with respect to small violation of symmetry (it causes a shift of such eigenvalues only). It is more interesting to look for eigenvalues (trapped modes) embedded in the continuous spectrum (above the threshold). These eigenvalues are unstable with respect to small violation of symmetry. In some cases it can remain an eigenvalue, but in other cases it can transform to a quasi-eigenvalue (resonance). Due to this effect, it has great influence on the transport properties of the system.

The existence of trapped modes are usually proved by the variational technique [15]. Numerical approaches are developed to compute these eigenvalues for some geometrical configurations (see, for example [20]). As for resonances, the situation is more difficult. The method of conformal map is used in [7] to obtain an order with respect to the width of the window for the distance between the resonance and the threshold. The asymptotics for the resonance was obtained in [14]. The resonance influence on electron transport can be used in nanodevices. For example, the system of three coupled waveguides (figure 1) can work as three-posed quantum switch [21]. Namely, let $d_1 - d_3$ be small, and we have an incoming wave in Ω^2 with the wavenumber close, for example, to the second threshold $4\pi^2/d_1^2$ of Ω^1. Then, we have resonance dependence of the transmission coefficient to each channel (see figure 2). It is easy to see that we can control the transmission to each channel by small variation of k. There is also another method of control. The position of the resonances (and, consequently, the position of the peaks in figure 2) depends on the width of the windows which change under the influence of bias voltage in a standard way (see, for example [1]). Note that the described resonance effects take place if there is a quasi-eigenvalue and not an eigenvalue. For the

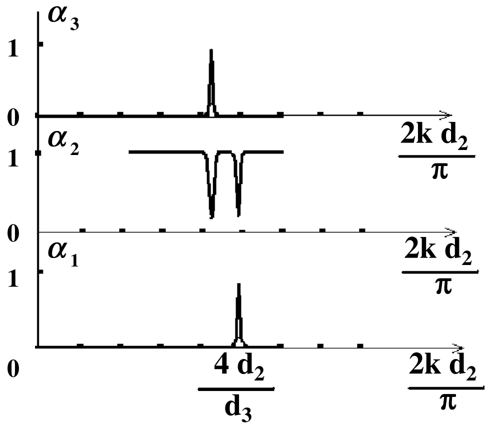

Figure 2. The dependence of the transmission coefficients $\alpha_1, \alpha_2, \alpha_3$ to Ω^1, Ω^2, Ω^3, respectively, on the wavenumber k near the second thresholds for the waveguides Ω^1, Ω^3. Dimensionless units are used.

device to work successfully, we should ensure that there is a small difference between d_1 and d_3. If $d_1 = d_3$ we have the eigenvalue. This is why it is important to look for the behaviour of the resonance for small $d_1 - d_3$. Our result gives this description. As for qualitative results, we show how small the order of difference $d_1 - d_3$ may be with respect to the width of the window $2a$. Namely, if $(d_1^2 - d_3^2)a^{-6} \to 0$ the quantum switch does not work successfully.

The described effect can be used for constructing a nanoelectronic device, electron trap. Consider three coupled waveguides with $d_1 \neq d_3$. Let an electron with energy close to the resonance k_a^2 come from infinity in Ω^2. Due to the existence of the resonance there will be an increase of electron density near the windows. We can vary the width of the quantum waveguides in a conventional way by changing the corresponding shift voltage using metal–oxide semiconductor (MOS) structures. Let us make d_3 equal to d_1. In this case we have a boundary state instead of the resonance. If it is made during a time less than the time of resonance decay (which is proportional to $1/\Im k_a^2$), we obtain the electron in the trap. To exclude the electron from the trap it is sufficient to break the geometrical symmetry again.

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